CHAPTER 4
MEASUREMENTS

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1 STANDARDS AND ACCURACY

1.1 Standards

Measurement is the process by which a quantitative comparison is made between a standard and a measurand. The measurand is the particular quantity of interest—the thing that is to be quantified. The standard of comparison is of the same character as the measurand, and so far as mechanical engineering is concerned the standards are defined by law and maintained by the National Institute of Standards and Technology (NIST, formerly known as the National Bureau of Standards). The four independent standards which have been defined are length, time, mass, and temperature.[1] All other standards are derived from these four. Before 1960 the standard for length was the international prototype meter, kept at Sevres, France. In 1960 the meter was redefined as 1,650,763.73 wavelengths of krypton light. Then in 1983, at the Seventeenth General Conference on Weights and Measures, a new standard was adopted: A meter is the distance traveled in a vacuum by light in 1/299,792,458 seconds.[2] However, there is a copy of the international prototype meter, known as the national prototype meter, kept by NIST. Below that level there are several bars known as national reference standards and below that there are the working standards. Interlaboratory standards in factories and laboratories are sent to NIST for comparison with the working standards. These interlaboratory standards are the ones usually available to engineers.

Standards for the other three basic quantities have also been adopted by NIST, and accurate measuring devices for those quantities should be calibrated against those standards.

The standard mass is a cylinder of platinum–iridium, the international kilogram, also kept at Sevres, France. It is the only one of the basic standards that is still established by a prototype. In the United States the basic unit of mass is the basic prototype kilogram No. 20. Working copies of this standard are used to determine the accuracy of interlaboratory standards. Force is not one of the fundamental quantities, but in the United States the standard unit of force is the pound, defined as the gravitational attraction for a certain platinum mass at sea level and 45° latitude.

Absolute time, or the time when some event occurred in history, is not of much interest to engineers. Engineers are more likely to need to measure time intervals, that is, the time

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between two events. The basic unit for time measurements is the *second*. At one time the second was defined as 1/86,400 of the average period of rotation of the earth on its axis, but that is not a practical standard. The period varies and the earth is slowing up. Consequently a new standard based on the oscillations associated with a certain transition within the cesium atom was defined and adopted. That standard, the cesium clock, has now been superceded by the cesium fountain atomic clock as the primary time and frequency standard of the United States.[3] Although this cesium ''clock'' is the basic frequency standard, it is not generally usable by mechanical engineers. Secondary standards such as tuning forks, crystals, electronic oscillators, and so on are used, but from time to time access to time standards of a higher order of accuracy may be required. To help meet these requirements, NIST broadcasts 24 hours per day, 7 days per week time and frequency information from radio stations WWV, WWVB, and WWVL located in Fort Collins, Colorado, and WWVH located in Hawaii. Other nations also broadcast timing signals. For details on the time signal broadcasts, potential users should consult NIST.[4]

Temperature is one of four fundamental quantities in the international measuring system. Temperature is fundamentally different in nature from length, time, and mass. It is an intensive quantity, whereas the others are extensive. Join together two bodies that have the same temperature and you will have a larger body at that same temperature. If you join two bodies which have a certain mass, you will have one body of twice the mass of the original body. Two bodies are said to be at the same temperature if they are in thermal equilibrium. The international practical temperature scale, adopted in 1990 (ITS-90) by the International Committee on Weights and Measurement is the one now in effect and the one with which engineers are primarily concerned. In this system the kelvin (K) is the basic unit of temperature. It is 1/273.16 of the temperature at the triple point of water, the temperature at which the solid, liquid, and vapor phases of water exist in equilibrium.[5] Degrees Celsius (°C) is related to degrees kelvin by the equation

$$t = T - 273.15$$

where t = degrees Celsius
T = degrees kelvin

1.2 Accuracy and Precision

In measurement practice four terms are frequently used to describe an instrument. They are accuracy, precision, sensitivity, and linearity. Accuracy, as applied to an instrument, is the closeness with which a reading approaches the true value. Since there is some error in every reading, the ''true value'' is never known. In the discussion of error analysis which follows, methods of estimating the ''closeness'' with which the determination of a measured value approaches the true value will be presented. Precision is the degree to which readings agree among themselves. If the same value is measured many times and all the measurements agree very closely, the instrument is said to have a high degree of precision. It may not, however, be a very accurate instrument. Accurate calibration is necessary for accurate measurement. Measuring instruments must, for accuracy, be from time to time compared to a standard. These will usually be laboratory or company standards which are in turn compared from time to time with a working standard at NIST. This chain can be thought of as the pedigree of the instrument, and the calibration of the instrument is said to be traceable to NIST.

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1.3 Sensitivity and Resolution

These two terms, as applied to a measuring instrument, refer to the smallest change in the measured quantity to which the instrument responds. Obviously the accuracy of an instrument will depend to some extent on the sensitivity. If, for example, the sensitivity of a pressure transducer is 1 kPa, any particular reading of the transducer has a potential error of at least 1 kPa. If the readings expected are in the range of 100 kPa and a possible error of 1% is acceptable, then the transducer with a sensitivity of 1 kPa may be acceptable, depending upon what other sources of error may be present in the measurement. A highly sensitive instrument is difficult to use. Therefore a sensitivity significantly greater than that necessary to obtain the desired accuracy is no more desirable than one with insufficient sensitivity.

Many instruments today have digital readouts. For such instruments the concepts of sensitivity and resolution are defined somewhat differently than they are for analog-type instruments. For example, the resolution of a digital voltmeter depends on the "bit" specification and the voltage range. The relationship between the two is expressed by the equation

$$R = \frac{V}{2^n}$$

where R = resolution in volts
V = voltage range
n = number of bits

Thus an 8-bit instrument on a 1-V scale would have a resolution of 1/256, or 0.004, volt. On a 10-V scale that would increase to 0.04 V. As in analog instruments, the higher the resolution, the more difficult it is to use the instrument, so if the choice is available, one should use the instrument which just gives the desired resolution and no more.

1.4 Linearity

The calibration curve for an instrument does not have to be a straight line. However, conversion from a scale reading to the corresponding measured value is most convenient if it can be done by multiplying by a constant rather than by referring to a nonlinear calibration curve or by computing from an equation. Consequently instrument manufacturers generally try to produce instruments with a linear readout, and the degree to which an instrument approaches this ideal is indicated by its *linearity*. Several definitions of linearity are used in instrument specification practice.[6] The so-called independent linearity is probably the most commonly used in specifications. For this definition the data for the instrument readout versus the input are plotted and then a "best straight line" fit is made using the method of least squares. Linearity is then a measure of the maximum deviation of any of the calibration points from this straight line. This deviation can be expressed as a percentage of the actual reading or a percentage of the full-scale reading. The latter is probably the most commonly used, but it may make an instrument appear to be much more linear than it actually is. A better specification is a combination of the two. Thus, linearity equals $+A$ percent of reading or $+B$ percent of full scale, whichever is greater. Sometimes the term independent linearity is used to describe linearity limits based on actual readings. Since both are given in terms of a fixed percentage, an instrument with A percent proportional linearity is much more accurate at low reading values than an instrument with A percent independent linearity.

It should be noted that although specifications may refer to an instrument as having A percent linearity, what is really meant is A percent nonlinearity. If the linearity is specified as independent linearity, the user of the instrument should try to minimize the error in

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readings by selecting a scale, if that option is available, such that the actual reading is close to full scale. A reading should never be taken near the low end of a scale if it can possibly be avoided.

For instruments that use digital processing, linearity is still an issue since the analog-to-digital converter used can be nonlinear. Thus linearity specifications are still essential.

2 IMPEDANCE CONCEPTS

Two basic questions which must be considered when any measurement is made are: How has the measured quantity been affected by the instrument used to measure it? Is the quantity the same as it would have been had the instrument not been there? If the answers to these questions are no, the effect of the instrument is called *loading.* To characterize the loading, the concepts of *stiffness* and *input impedance* are used.[7] At the input of each component in a measuring system there exists a variable q_{i1} which is the one we are primarily concerned with in the transmission of information. At the same point, however, there is associated with q_{i1} another variable q_{i2} such that the product $q_{i1}q_{i2}$ has the dimensions of power and represents the rate at which energy is being withdrawn from the system. When these two quantities are identified, the generalized input impedance Z_{gi} can be defined by

$$Z_{gi} = \frac{q_{i1}}{q_{i2}} \tag{1}$$

if q_{i1} is an *effort variable.* The effort variable is also sometimes called the *across variable.* The quantity q_{i2} is called the *flow variable* or *through variable.* In the dynamic case these variables can be represented in the frequency domain by their Fourier transform. Then the quantity Z is a complex number.The application of these concepts is illustrated by the example in Fig. 1. The output of the linear network in the blackbox (Fig. 1*a*) is the open-circuit voltage E_o until the load Z_L is attached across the terminals A–B. If Thévenin's theorem is applied after the load Z_L is attached, the system in Fig. 1*b* is obtained. For that system the current is given by

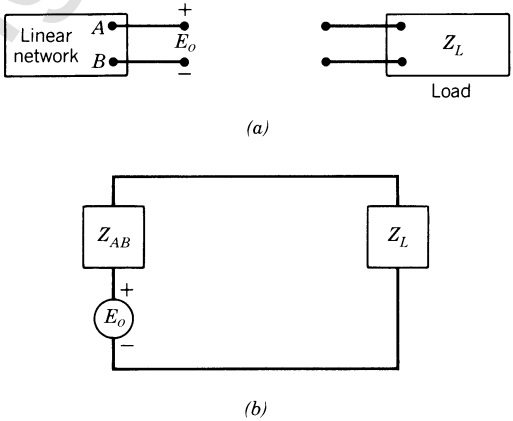


Figure 1 Application of Thévenin's theorem.

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$$i_m = \frac{E_o}{Z_{AB} + Z_L} \tag{2}$$

and the voltage E_L across Z_L is

$$E_L = i_m Z_L = \frac{E_o Z_L}{Z_{AB} + Z_L}$$

or

$$E_L = \frac{E_o}{1 + Z_{AB}/Z_L} \tag{3}$$

Equations (2)–(7) are frequency-domain equations.

In a measurement situation E_L would be the voltage indicated by the voltmeter, Z_L would be the input impedance of the voltmeter, and Z_{AB} would be the output impedance of the linear network. The true output voltage, E_o, has been reduced by the voltmeter, but it can be computed from the voltmeter reading if Z_{AB} and Z_L are known. From Eq. (3) it is seen that the effect of the voltmeter on the reading is minimized by making Z_L as large as possible.

If the generalized input and output impedances Z_{gi} and Z_{go} are defined for nonelectrical systems as well as electrical systems, Eq (3) can be generalized to

$$q_{im} = \frac{q_{iu}}{1 + Z_{go}/Z_{gi}} \tag{4}$$

where q_{im} is the measured value of the effort variable and q_{iu} is the undisturbed value of the effort variable. The output impedance Z_{go} is not always defined or easy to determine; consequently Z_{gi} should be large. If it is large enough, knowing Z_{go} is unimportant.

If q_{i1} is a flow variable rather than an effort variable (current is a flow variable, voltage an effort variable), it is better to define an input admittance

$$Y_{gi} = \frac{q_{i1}}{q_{i2}} \tag{5}$$

rather than the generalized input impedance

$$Z_{gi} = \frac{\text{effort variable}}{\text{flow variable}}$$

The power drain of the instrument is

$$P = q_{i1} q_{i2} = \frac{q_{i2}^2}{Y_{gi}} \tag{6}$$

Hence, to minimize power drain, Y_{gi} must be large. For an electrical circuit

$$I_m = \frac{I_u}{1 + Y_o/Y_i} \tag{7}$$

where I_m = measured current
I_u = actual current
Y_o = output admittance of circuit
Y_i = input admittance of meter

When the power drain is zero and the deflection is zero, as in structures in equilibrium, for example when deflection is to be measured, the concepts of impedance and admittance are

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replaced with the concepts of *static stiffness* and *static compliance*. Consider the idealized structure in Fig. 2.

To measure the force in member K_2, an elastic link with a spring constant K_m is inserted in series with K_2. This link would undergo a deformation proportional to the force in K_2. If the link is very soft in comparison with K_1, no force can be transmitted to K_2. On the other hand, if the link is very stiff, it does not affect the force in K_2 but it will not provide a very good measure of the force. The measured variable is an effort variable, and in general, when it is measured, it is altered somewhat. To apply the impedance concept, a flow variable whose product with the effort variable gives power is selected. Thus,

$$\text{Flow variable} = \frac{\text{power}}{\text{effort variable}}$$

Mechanical impedance is then defined as force divided by velocity, or

$$Z = \frac{\text{force}}{\text{velocity}}$$

where force and velocity are dynamic quantities represented by their Fourier transform and Z is a complex number. This is the equivalent of electrical impedance. However, if the static mechanical impedance is calculated for the application of a constant force, the impossible result

$$Z = \frac{\text{force}}{0} = \infty$$

is obtained.

This difficulty is overcome if energy rather than power is used in defining the variable associated with the measured variable. In that case the static mechanical impedance becomes the *stiffness:*

$$\text{Stiffness} = S_g = \frac{\text{effort}}{\int \text{flow}\, dt}$$

In structures,

$$S_g = \frac{\text{effort variable}}{\text{displacement}}$$

When these changes are made, the same formulas used for calculating the error caused by the loading of an instrument in terms of impedances can be used for structures by inserting S for Z. Thus

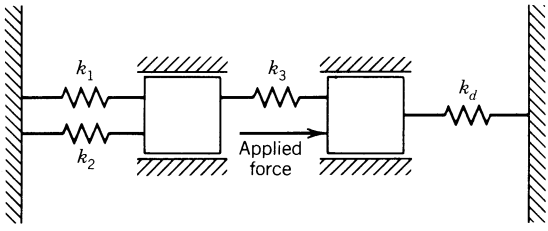


Figure 2 Idealized elastic structure.

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$$q_{im} = \frac{q_{iu}}{1 + S_{go}/S_{gi}} \tag{8}$$

where q_{im} = measured value of effort variable
q_{iu} = undisturbed value of effort variable
S_{go} = static output stiffness of measured system
S_{gi} = static stiffness of measuring system

For an elastic force-measuring device such as a load cell S_{gi} is the spring constant K_m. As an example, consider the problem of measuring the reactive force at the end of a propped cantilever beam, as in Fig. 3.

According to Eq. (8), the force indicated by the load cell will be

$$F_m = \frac{F_u}{1 + S_{go}/S_{gi}}$$

$$S_{gi} = K_m \qquad \text{and} \qquad S_{go} = \frac{3EI}{L^3}$$

The latter is obtained by noting that the deflection at the tip of a tip-loaded cantilever is given by

$$\delta = \frac{PL^3}{3EI}$$

The stiffness is the quantity by which the deflection must be multiplied to obtain the force producing the deflection.

For the cantilever beam

$$F_m = \frac{F_u}{1 + 3EI/K_mL^3} \tag{9}$$

or

$$F_u = F_m\left(1 + \frac{3EI}{K_mL^3}\right) \tag{10}$$

Clearly, if $K_m >> 3EI/L^3$, the effect of the load cell on the measurement will be negligible.

To measure displacement rather than force, the concept of compliance is introduced and defined as

$$C_g = \frac{\text{flow variable}}{\int \text{effort variable } dt}$$

Then

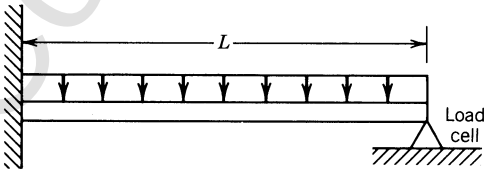


Figure 3 Measuring the reactive force at the tip.

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$$q_m = \frac{q_u}{1 + C_{go}/C_{gi}} \tag{11}$$

If displacements in an elastic structure are considered, the compliance becomes the reciprocal of stiffness, or the quantity by which the force must be multiplied to obtain the displacement caused by the force. The cantilever beam in Fig. 4 again provides a simple illustrative example.

If the deflection at the tip of this cantilever is to be measured using a dial gage with a spring constant K_m,

$$C_{gi} = \frac{1}{K_m} \qquad \text{and} \qquad C_{go} = \frac{L^3}{3EI}$$

Thus

$$\delta_m = \delta_u\left(1 + \frac{K_m L^3}{3EI}\right) \tag{12}$$

Not all interactions between a system and a measuring device lend themselves to this type of analysis. A pitot tube, for example, inserted into a flow field distorts the flow field but does not extract energy from the field. Impedance concepts cannot be used to determine how the flow field will be affected.

There are also applications in which it is not desirable for a force-measuring system to have the highest possible stiffness. A subsoil pressure gage is an example. Such a gage, if it is much stiffer than the surrounding soil, will take a disproportionate share of the total load and will consequently indicate a higher pressure than would have existed in the soil if the gage had not been there.

3 ERROR ANALYSIS

It may be accepted as axiomatic that there will always be errors in measured values. Thus if a quantity X is measured, the correct value q, and X will differ by some amount e. Hence

$$\pm(q - X) = e$$

or

$$q = X \pm e \tag{13}$$

It is essential, therefore, in all measurement work that a realistic estimate of e be made. Without such an estimate the measurement of X is of no value. There are two ways of

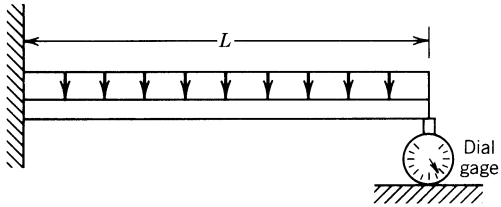


Figure 4 Measuring the tip deflection.

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estimating the error in a measurement. The first is the external estimate, or $\mathscr{E}_E$, where $\mathscr{E} = e/q$. This estimate is based on knowledge of the experiment and measuring equipment and to some extent on the internal estimate $\mathscr{E}_I$.

The internal estimate is based on an analysis of the data using statistical concepts.

3.1 Internal Estimates

If a measurement is repeated many times, the repeat values will not, in general, be the same. Engineers, it may be noted, do not usually have the luxury of repeating measurements many times. Nevertheless the standardized means for treating results of repeated measurements are useful, even in the error analysis for a single measurement.[8]

If some quantity is measured many times and it is assumed that the errors occur in a completely random manner, that small errors are more likely to occur than large errors, and that errors are just as likely to be positive as negative, the distribution of errors can be represented by the curve

$$F(X) = \frac{Y_o e^{-(X-U)}}{2\sigma^2} \tag{14}$$

where $F(X)$ = number of measurements for a given value of $(X - U)$
Y_o = maximum height of curve or number of measurements for which $X = U$
U = value of X at point where maximum height of curve occurs σ determines lateral spread of the curve

This curve is the normal, or Gaussian, frequency distribution. The area under the curve between X and δX represents the number of data points which fall between these limits and the total area under the curve denotes the total number of measurements made. If the normal distribution is defined so that the area between X and $X + \delta X$ is the probability that a data point will fall between those limits, the total area under the curve will be unity and

$$F(X) = \frac{\exp -(X - U)^2/2\sigma^2}{\sigma\sqrt{2\Pi}} \tag{15}$$

and

$$P_x = \int \frac{\exp -(X - U)^2/2\sigma^2}{\sigma\sqrt{2\Pi}}\, dx \tag{16}$$

Now if U is defined as the average of all the measurements and s as the standard deviation,

$$\sigma = \left[\frac{\Sigma(X - U)^2}{N}\right]^{1/2} \tag{17}$$

where N is the total number of measurements. Actually this definition is used as the best estimate for a universe standard deviation, that is, for a very large number of measurements. For smaller subsets of measurements the best estimate of σ is given by

$$\sigma = \left(\frac{\Sigma(X - U)^2}{n - 1}\right)^{1/2} \tag{18}$$

where n is the number of measurements in the subset. Obviously the difference between the two values of σ becomes negligible as n becomes very large (or as $n \rightarrow N$).

The probability curve based on these definitions is shown in Fig. 5.

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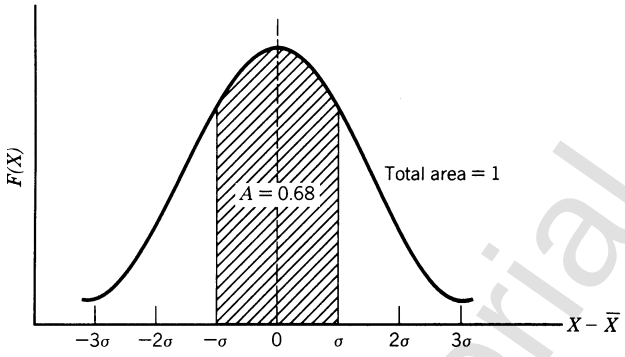


Figure 5 Probability curve.

The area under this curve between $-\sigma$ and $+\sigma$ is 0.68. Hence 68% of the measurements can be expected to have errors that fall in the range of $\pm\delta$. Thus the chances are 68/32, or better than 2 to 1, that the error in a measurement will fall in this range. For the range $\pm 2\sigma$ the area is 0.95. Hence 95% of all the measurement errors will fall in this range and the odds are about 20:1 that a reading will be within this range. The odds are about 384:1 that any given error will be in the range of $\pm 3\sigma$.

Some other definitions related to the normal distribution curve are as follows:

1. *Probable Error.* The error likely to be exceeded in half of all the measurements and not reached in the other half of the measurements. This error in Fig. 5 is about 0.67δ.
2. *Mean Error.* The arithmetic mean of all the errors regardless of sign. This is about 0.8σ.
3. *Limit of Error.* The error that is so large it is most unlikely ever to occur. It is usually taken as 4δ.

3.2 Use of Normal Distribution to Calculate Probable Error in X

The foregoing statements apply strictly only if the number of measurements is very large. Suppose that n measurements have been made. That is a sample of n data points out of an infinite number. From that sample U and σ are calculated as above. How good are these numbers? To determine that, we proceed as follows: Let

$$U = F(X_1, X_2, X_3, \ldots, X_n) = \frac{\sum X_i}{n} \tag{19}$$

$$e_u = \sum \frac{\partial F}{\partial X_i} e_{xi} \tag{20}$$

where e_u = error in U
e_{xi} = error in X_i

$$(e_u)^2 = \sum \left(\frac{\partial F}{\partial X_i} e_{xi} \right)^2 + \sum \left(\frac{\partial F}{\delta X_i} e_{xi} \right) \left(\frac{\delta F}{\delta X_j} e_{xj} \right) \tag{21}$$

where $I \neq j$. If the errors e_i to e_n are independent and symmetrical, the cross-product terms will tend to disappear and

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$$(e_u)^2 = \Sigma\left(\frac{\partial F}{\partial X_i}\, e_{xi}\right)^2 \tag{22}$$

Since $\partial F/\partial X_i = 1/n$,

$$e_u = \left[\Sigma \left(\frac{1}{n}\right)^2 e_{xi}^2\right]^{1/2} \tag{23}$$

or

$$e_u = \left[\left(\frac{1}{n}\right)^2 \Sigma(e_{xi})^2\right]^{1/2} \tag{24}$$

from the definition of σ

$$\Sigma(e_{xi})^2 = n\sigma^2 \tag{25}$$

and

$$e_u = \frac{\sigma}{\sqrt{n}}$$

This equation must be corrected because the real errors in X are not known. If the number n were to approach infinity, the equation would be correct. Since n is a finite number, the corrected equation is written as

$$e_u = \frac{\sigma}{(n-1)^{1/2}} \tag{26}$$

and

$$q = U \pm \frac{\sigma}{(n-1)^{1/2}} \tag{27}$$

This says that if one reading is likely to differ from the true value by an amount σ, then the average of 10 readings will be in error by only $\sigma/3$ and the average of 100 readings will be in error by $\sigma/10$. To reduce the error by a factor of 2, the number of readings must be increased by a factor of 4.

3.3 External Estimates

In almost all experiments several steps are involved in making a measurement. It may be assumed that in each measurement there will be some error, and if the measuring devices are adequately calibrated, errors are as likely to be positive as negative. The worst condition insofar as accuracy of the experiment is concerned would be for all errors to have the same sign. In that case, assuming the errors are all much less than 1, the resultant error will be the sum of the individual errors, that is,

$$€_E = €_1 + €_2 + €_3 + \cdots \tag{28}$$

It would be very unusual for all errors to have the same sign. Likewise it would be very unusual for the errors to be distributed in such a way that

$$€_E = 0$$

A general method follows for treating problems that involve a combination of errors to determine what error is to be expected as a result of the combination.

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Suppose that

$$V = F(a, b, c, d, e, \cdots, x, y, z) \tag{29}$$

where $a, b, c, \cdots x, y, z$ represent quantities which must be individually measured to determine V. Then

$$\delta V = \Sigma \left(\frac{\partial F}{\partial n}\right) \delta n$$

and

$$\text{€}_E = \Sigma \left(\frac{\partial F}{\partial n}\right) e_n \tag{30}$$

The sum of the squares of the error contributions is given by

$$e_E^2 = \left[\Sigma \left(\frac{\partial F}{\partial n}\right) e_n\right]^2 \tag{31}$$

Now, as in the discussion of internal errors, assume that errors e_n are independent and symmetrical. This justifies taking the sum of the cross products as zero:

$$\Sigma \left(\frac{\partial F}{\partial n}\right)\left(\frac{\partial F}{\partial m}\right) e^n e^m = 0 \qquad n \neq m \tag{32}$$

Hence

$$(\text{€}_E)^2 = \Sigma \left(\frac{\partial F}{\partial n}\right)^2 e_n^2$$

or

$$e_E = \left[\Sigma \left(\frac{\partial F}{\partial n}\right)^2 e_n^2\right]^{1/2} \tag{33}$$

This is the *most probable value* of e_E. It is much less than the worst case:

$$\text{€}_e = [|\text{€}_a| + |\text{€}_b| + |\text{€}_c| \cdots + |\text{€}_z|] \tag{34}$$

As an application, the determination of g, the local acceleration of gravity, by use of a simple pendulum will be considered:

$$g = \frac{4\Pi^2 L}{T^2} \tag{35}$$

where L = length of pendulum
T = period of pendulum

If an experiment is performed to determine g, the length L and the period T would be measured. To determine how the accuracy of g will be influenced by errors in measuring L and T write

$$\frac{\partial g}{\partial L} = \frac{4\Pi^2}{T^2} \qquad \text{and} \qquad \frac{\partial g}{\partial T} = \frac{-8\Pi^2 L}{T^3} \tag{36}$$

The error in g is the variation in g written as follows:

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$$\delta g = \left(\frac{\partial g}{\partial L}\right) \Delta L + \left(\frac{\partial g}{\partial T}\right) \Delta T \tag{37}$$

or

$$\delta g = \left(\frac{4\Pi^2}{T^2}\right) \Delta L - \left(\frac{8\Pi^2 L}{T^3}\right) \Delta T \tag{38}$$

It is always better to write the errors in terms of percentages. Consequently Eq. (38) is rewritten as

$$\delta g = \frac{(4\Pi^2 L/T^2)\ \Delta \mathrm{L}}{L} - \frac{2(4\Pi^2 L/T^2)\ \Delta T}{T} \tag{39}$$

or

$$\frac{\delta g}{g} = \frac{\Delta L}{L} - \frac{2\ \Delta T}{T} \tag{40}$$

Then

$$e_g = [e_L^2 + (2e_T)^2]^{1/2} \tag{41}$$

where e_g is the *most probable error* in the measured value of g. That is,

$$g = \frac{4\Pi^2 L}{T^2} \pm e_g \tag{42}$$

where L and T are the measured values. Note that even though a positive error in T causes a negative error in the calculated value of g, the contribution of the error in T to the most probable error is taken as positive. Note also that an error in T contributes four times as much to the most probable error as an error in L contributes. It is fundamental in measurements of this type that those quantities which appear in the functional relationship raised to some power greater than unity contribute more heavily to the most probable error than other quantities and must, therefore, be measured with greater care.

The determination of the most probable error is simple and straightforward. The question is how are the errors, such as $\Delta L/L$ and $\Delta T/T$, determined. If the measurements could be repeated often enough, the statistical methods discussed in the internal error evaluation could be used to arrive at a value. Even in that case it would be necessary to choose some representative error such as the standard deviation or the mean error. Unfortunately, as was noted previously, in engineering experiments it usually is not possible to repeat measurements enough times to make statistical treatments meaningful. Engineers engaged in making measurements will have to use what knowledge they have of the measuring instruments and the conditions under which the measurements are made to make a reasonable estimate of the accuracy of each measurement. When all of this has been done and a most probable error has been calculated, it should be remembered that the result is not the actual error in the quantity being determined but is, rather, the engineer's best estimate of the magnitude of the uncertainty in the final result.[9,10]

Consider again the problem of determining g. Suppose that the length L of the pendulum has been determined by means of a meter stick with 1-mm calibration marks and the error in the calibration is considered negligible in comparison with other errors. Suppose the value of L is determined to be 91.7 cm. Since the calibration marks are 1 mm apart, it can be assumed that ΔL is no greater than 0.5 mm. Hence the maximum

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$$\frac{\Delta L}{L} = 5.5 \times 10^{-4}$$

Suppose T is determined with the pendulum swinging in a vacuum with an arc of $\pm 5°$ using a stop watch that has an inherent accuracy of one part in 10,000. (If the arc is greater than $\pm 5°$, a nonisochronous swing error enters the picture.) This means that the error in the watch reading will be no more than 10^{-4} s. However, errors are introduced in the period determination by human error in starting and stopping the watch as the pendulum passes a selected point in the arc. This error can be minimized by selecting the highest point in the arc because the pendulum has zero velocity at that point and timing a large number of swings so as to spread the error out over that number of swings. Human reaction time may vary from as low as 0.2 s to as high as 0.7 s. A value of 0.5 s will be assumed. Thus the estimated maximum error in starting and stopping the watch will be 1 s (± 0.5 s at the start and ± 0.5 s at the stop). A total of 100 swings will be timed. Thus the estimated maximum error in the period will be 1/100 s. If the period is determined to be 1.92 s, the estimated maximum error will be $0.01/1.92 = 0.005$. Compared to this, the error in the period due to the inherent inaccuracy of the watch is negligible. The nominal value of g calculated from the measured values of L and T is 982.03 cm/s^2. The most probable error [Eq. (29)] is

$$[4(0.005)^2 + (5.5 \times 10^{-4})^2]^{1/2} = 0.01 \tag{43}$$

The uncertainty in the value of g is then ± 9.82 cm/s^2, or in other words the value of g will be somewhere between 972.21 and 991.85 cm/s^2.

Often it is necessary for the engineer to determine in advance how accurately the measurements must be made in order to achieve a given accuracy in the final calculated result. For example, in the pendulum problem it may be noted that the contribution of the error in T to the most probable error is more than 300 times the contribution of the error in the length measurement. This suggests, of course, that the uncertainty in the value of g could be greatly reduced if the error in T could be reduced. Two possibilities for doing this might be (1) find a way to do the timing that does not involve human reaction time or (2) if that is not possible, increase the number of cycles timed. If the latter alternative is selected and other factors remain the same, the error in T timed over 200 swings is 1/200 or 0.005, second. As a percentage the error is $0.005/1.92 = 0.0026$. The most probable error in g then becomes

$$e_g = [4 \times (2.6 \times 10^{-3})^2 + (5.5 \times 10^{-4})^2]^{1/2} = 0.005 \tag{44}$$

This is approximately half of the most probable error in the result obtained by timing just 100 swings. With this new value of e_g the uncertainty in the value of g becomes ± 4.91 cm/s^2 and g then can be said to be somewhere between 977.12 and 986.94 cm/s^2. The procedure for reducing this uncertainty still further is now self-evident.

Clearly the value of this type of error analysis depends upon the skill and objectivity of the engineer in estimating the errors in the individual measurements. Such skills are acquired only by practice and careful attention to all the details of the measurements.

REFERENCES

1. W. A. Wildhack, "NBS Source of American Standards," *ISA Journal* **8**(2) (February 1961).
2. P. Giacomo "News from the IBPM," *Metrologia* **20**(1) (April, 1984).
3. NIST-F1 Cesium Fountain Atomic Clock.

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4. ''NIST Time and Frequency Services,'' NIST Special Publication 432, 2002.

5. R. E. Bentley (ed.) *Handbook of Temperature Measurement,* CSIRO Springer.

6. E. A. Doebelin, *Measurement Systems—Application and Design,* 5th ed., McGraw Hill, New York, 2004, pp. 85–91.

7. C. M. Harris and A. G. Piersol, ''Mechanical Impedance,'' in *Shock and Vibration Handbook,* 5th ed. McGraw-Hill, New York, 2002, Chapter 10, pp. 10.1–10.14.

8. N. H. Cook and E. Rabinowicz, *Physical Measurement and Analysis,* Addison Wesley, Readng, MA, 1963, pp. 29–68.

9. S. J. Kline and F. A. McClintock, ''Describing Uncertainties in Single Sample Experiments,'' *Mechanical Engineering,* **75**(3) (January 1953).

10. B. N. Taylor and C. E. Kuyatt, ''Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results,'' NIST Technical Note 1297, 1994.

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